

# Waves propagation in turbulent superfluid helium in presence of combined rotation and counterflow

R. A. Peruzza

M. Sciacca\*

*Dipartimento di Metodi e Modelli Matematici Università di Palermo,  
Facoltà di Ingegneria, Viale delle Scienze, 90128 Palermo, Italy*

## Abstract

A complete study of the propagation of waves (namely longitudinal density and temperature waves, longitudinal and transversal velocity waves and heat waves) in turbulent superfluid helium is made in three situations: a rotating frame, a thermal counterflow, and the simultaneous combination of thermal counterflow and rotation. Our analysis aims to obtain as much as possible information on the tangle of quantized vortices from the wave speed and attenuation factor of these different waves, depending on their relative direction of propagation with respect to the rotation vector.

*PACS number(s):* 05.70.Ln, 67.40.Vs, 67.40.Bz, 47.27.2i

*A.M.S. Classification:* 82D50

*Key words:* non equilibrium thermodynamics; superfluid turbulence; second sound; vortex tangle

## 1 Introduction

The most known phenomenological model, accounting for many of the properties of He II, given by *Tisza* [1] and *Landau* [2] is called the two-fluid model. The basic assumption is that the liquid behaves as a mixture of two fluids: the normal component with density  $\rho_n$  and velocity  $\mathbf{v}_n$ , and the superfluid component with density  $\rho_s$  and velocity  $\mathbf{v}_s$ . When the difference  $\mathbf{V} := \mathbf{v}_n - \mathbf{v}_s$  between the normal and superfluid velocities, known also as counterflow velocity, exceed a certain critical velocity, a *mutual friction*  $\mathbf{F}_{sn}$  has to be included. This friction force is attributed to an interaction of the normal component with the vortices in the superfluid.

Quantized superfluid vortices play an important role in the hydrodynamics of the fluid and they have been the object of many studies. The state of the fluid in which vortices are present, is referred to as the *superfluid turbulent state*. A review on superfluid turbulence can be found in Tough's paper [3] and in chapter 7 of Donnelly's book [4]. The quantized vortices created by applying a thermal counterflow form an irregular, spatially disordered tangle of lines. In this case, the vortex line density  $L$  (length of vortex line per unit volume) is  $L_H \approx \gamma^2 V^2$ , where  $\mathbf{V}$  is the modulus of the relative velocity between the two components of the mixture and  $\gamma$  a temperature-dependent coefficient [3]. The vortex system is almost isotropic, provided that one neglects a small anisotropy induced by the imposed counterflow [5].

---

\*Corresponding author.

<sup>0</sup>E-mail addresses: peruzza@unipa.it (R.A. Peruzza), msciacca@unipa.it (M. Sciacca)

The creation of the vortices cannot be made only in this way; in fact, the first studies of quantized vorticity involved a sample of He II rotating at constant angular velocity  $\Omega$  exceeding a certain small critical value. The results brought to an ordered array of vortices aligned along the rotation axis, whose number density per unit area is given by Feynman's rule  $L_R = 2\Omega/\kappa$ , where  $\kappa = h/m = 9.97 \cdot 10^{-4} \text{ cm}^2/\text{sec}$  is the quantum of circulation, with  $h$  Planck's constant and  $m$  mass of the helium atom.

Now, an important question naturally arises: what happens if vortices are created by both rotation and counterflow? There has been only one experiment of which we are aware [6], on the formation of vortices in combined rotation and counterflow along the rotational axis. This experiment suggests that there exists a form of steady rotating turbulence, characterized by a vortex line density at given counterflow velocity  $\mathbf{V}$  and angular velocity  $\mathbf{\Omega}$ . Swanson et al. [6] found that at slow rotation the critical counterflow velocity above which the flow became turbulent was greatly reduced. The experimental observations showed that the two effects (thermal counterflow and rotation) are not merely additive, in fact for  $V$  high the measured values of  $L$  are always less than  $L_H + L_R$ . However, from our point of view the results of these experiments are purely qualitative because the authors didn't take the anisotropy of the vortex tangle in consideration which, as we will see in the last section of this paper, is essential to know the spatial distribution of the vortex tangle in liquid Helium II through measurements of second sound attenuation.

The aim of this work is to study the propagation of longitudinal density and temperature waves, and longitudinal and transversal velocity waves and heat waves in the combined situation of a rotating frame and of a cylindrical container in presence of thermal counterflow. The studies of the two separate cases of pure rotation and pure thermal counterflow are also considered in order to give a more complete view of the wave propagation in these three different situations. The influence of the parameters characterizing the vorticity on the propagation of the waves is shown explicitly. The practical interest of this research is to obtain information on the vortex tangle from measurements on wave propagation. This is an important issue, because under the combined influence of rotation and counterflow the vortex tangle cannot be assumed isotropic. Then, we must find not only the vortex line density  $L$  but also the geometrical characterization, which requires, in principle, to consider wave propagation in different directions, as well as a deeper full analysis of waves. Note that here the tangle itself is not considered as a dynamical quantity, because it is not modified by the second sound. For this reason, evolution equations for the tangle are not needed here.

The plan of this paper is the following: Section 2 is concerned with the model for helium II, in which the use of a pressure tensor associated to the vorticity has been considered; in Section 3 and in Section 4 we study wave propagation in rotating frame and in presence of thermal counterflow respectively, pure rotation is analyzed in the general case in which a component of the mutual friction force parallel to the rotation axis is present; finally, in Section 5 we study wave propagation in simultaneous rotation and counterflow, analyzing two different situations about the relative direction of wave propagation with respect to the rotation vector.

## 2 Evolution equations

Many observations have shown that both thermal conductivity  $\lambda_1$  and the relaxation time of the heat flux  $\tau_1$  in helium II are very high. As observed in [7] their ratio  $\frac{\lambda_1}{\tau_1} := \zeta < \infty$  determines the velocity of the second sound, which is a heat wave propagating in the superfluid.

As a consequence, it is natural to use a thermodynamical theory where the heat flux  $\mathbf{q}$  appears as a further fundamental field. In this way, a linear macroscopic one-fluid model of liquid helium II, based on Extended Thermodynamics [8, 9], has been formulated [7]. This model is able to describe the laminar flow of the superfluid both in the presence and in absence of dissipative phenomena and to predict the propagation of the two sounds in bulk liquid helium II and of the fourth sound in liquid helium flowing in a porous medium [7], [10]-[13], in agreement with microscopic and experimental data.

In order to describe the presence of vortices in rotating helium II, in superfluid turbulence or in combined rotation and thermal counterflow, the use of a further additional pressure tensor  $\mathbf{P}_\omega$ , associated to the vorticity, is necessary. The simplified situations of a rotating frame and of pure thermal counterflow have been considered in [14], where a constitutive relation for  $\mathbf{P}_\omega$  and its influence on the dynamics of the heat flux has been studied.

In this work the more complex situation involving thermal counterflow in a rotating cylinder, which is receiving much attention recently [15]-[21], is considered too. We start from a linear macroscopic one-fluid model of liquid helium II, whose fundamental fields are the density  $\rho$ , the barycentric velocity  $\mathbf{v}$ , which is related to the two velocities of the two-fluid model by the relation  $\rho\mathbf{v} = \rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n$ , the temperature  $T$  and the heat flux  $\mathbf{q}$ , related to the counterflow velocity  $\mathbf{V}$  by the relation  $\mathbf{q} = \rho_s T s \mathbf{V}$  (where  $s$  is the entropy of the Helium II). In the two-fluid model the natural variables are  $\mathbf{v}_s$  and  $\mathbf{v}_n$ , but in the experiments it is  $\mathbf{v}$  and  $\mathbf{q}$  which are directly measured. Therefore, the use of  $\mathbf{v}$  and  $\mathbf{q}$  appears suitable for our analysis. Neglecting the bulk and shear viscosity and under the hypothesis of small thermal dilatation (which in helium II are indeed very small), the linearized system of field equations for liquid helium II, in a non inertial frame, in absence of external force, is [14]:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_i}{\partial x_i} = 0 \\ \rho \frac{\partial v_i}{\partial t} + \frac{\partial p}{\partial x_i} + \mathbf{i}_i^0 + 2\rho (\boldsymbol{\Omega} \wedge \mathbf{v})_i = 0 \\ \frac{\partial T}{\partial t} + \frac{1}{\rho c_V} \frac{\partial q_j}{\partial x_j} = 0 \\ \frac{\partial q_i}{\partial t} + \zeta \frac{\partial T}{\partial x_i} + 2(\boldsymbol{\Omega} \wedge \mathbf{q})_i = (\vec{\sigma}_\omega)_i = -(\mathbf{P}_\omega \cdot \mathbf{q})_i \end{cases} \quad (2.1)$$

In this system,  $\mathbf{i}^0 + 2\rho(\boldsymbol{\Omega} \wedge \mathbf{v})$  is the inertial force,  $\zeta$  is a positive coefficient linked to the second sound velocity, and:

$$p = p_E(\rho, T) \quad \text{and} \quad c_V = \left( \frac{\partial \epsilon(\rho, T)}{\partial T} \right)_\rho \quad (2.2)$$

are the thermostatic pressure and the specific heat respectively ( $\epsilon$  is the specific internal energy). The effect of vortices is described by incorporating the source term  $\mathbf{P}_\omega \cdot \mathbf{q}$  to the evolution equation of the heat flux. As we will see, the expression of  $\mathbf{P}_\omega$  will assume different expressions in the different situations considered.

Now, a small comparison between the one-fluid model and the two-fluid model could be useful. With the corresponding transformations between the natural variables in the one fluid model,  $\mathbf{v}$  and  $\mathbf{q}$ , and those in the two-fluid model,  $\mathbf{v}_s$  and  $\mathbf{v}_n$ , the evolution equations (2.1) of the one-fluid model are equivalent to those of the two-fluid model in the linear approximation [14]. A formal difference is found in the form of the production term in the evolution equation for the heat flux (2.1d). When specified to pure rotation, this production term, as given by (3.2), has the usual Hall-Vinen form, whereas when specified to counterflow, the production term, as given by (4.1), yields the well-known Gorter-Mellinck form. These two situations have

been well explored in the context of the two-fluid and one-fluid frameworks. In the combined situation with simultaneous rotation and counterflow, the general form of the production term in (2.1d) is especially useful, as expressed in (5.1) and (5.2), because it allows one to write in an explicit and appealing way the anisotropy of the tangle, whose influence on the second sound is one of our main concerns. Given the same geometrical conditions for the tangle — which here are given a priori, and whose form is probed by means of second sound —, the evolution equations of the two-fluid model would coincide with those of the one-fluid model. Thus, the dispersion relations obtained here should be valid also in the context of the two-fluid model.

The one-fluid and the two-fluid models are not identical to each other. However, their mutual differences arise in contexts which are not relevant in the analysis presented here. For instance, one difference arises in the fourth sound in helium through porous media, in which some experimental results seem to support the one-fluid model [22]. Anyway, the two-fluid model could also cope with that situation provided the assumption that the superfluid component carries no entropy is slightly relaxed by assuming that it may carry a small but nonvanishing entropy. Other differences arise concerning the interaction between second sound and the vortex tangle. Here, we have assumed that second sound does not modify the vortex line density nor the geometrical structure of the tangle. If it is assumed that it may modify the vortex tangle, more general evolution equations would be needed, as for instance an evolution equation for the vortex line density  $L$  coupled with the rotation and counterflow, which have already been explored in the literature [23]. For instance, the evolution equation for  $L$  could be different — a generalized form of Vinen’s equation with the mentioned couplings has been proposed and studied [16] — but this is not relevant here because an equation for  $L$  is not necessary in this paper, as  $L$  is taken as fixed, and its value must be found from wave experiments. Some other differences may appear, concerning, for instance, the possibility of vortex density waves at high frequencies in the one-fluid model [24] that do not arise in the Hall-Vinen-Bekarevich-Khalatnikov model [25]. Since in this work we are focusing our attention to a situation in which the interaction between the second sound and the tangle does not distort the vortex lines nor the vortex density, the dispersion relations obtained in this paper by using the production terms  $\mathbf{P}_\omega(\mathbf{q}, \boldsymbol{\Omega}) \cdot \mathbf{q}$  would be also valid in the two-fluid context by using a production term of the form  $\mathbf{P}_\omega(\mathbf{v}_n - \mathbf{v}_s, \boldsymbol{\Omega}) \cdot (\mathbf{v}_n - \mathbf{v}_s)$  in an evolution equation for the relative velocity  $\mathbf{V} = \mathbf{v}_n - \mathbf{v}_s$ .

### 3 Wave propagation in rotating frame

We generalize here the results of [14] to the case in which a small interaction between second sound and vortex line parallel to the rotation axis is present. In [26], Hall and Vinen described experiments of liquid Helium II in a rotating frame, showing the main effects on the propagation and attenuation of the second sound as a consequence of the interaction between quasi-particles and vortex lines: these interactions are mainly present in the planes orthogonal to the rotation axis. As consequence of these experiments, in [14], Jou, Lebon and Mongiovì proposed an expression for the production term  $\vec{\sigma}_\omega$  in (2.1d), which takes into account dissipative and non dissipative contributions of the interaction between quasi-particles and vortex lines, but they did not consider interactions parallel to the rotation axis.

In another experiment [27], Snyder studied the component of mutual friction along the rotational axis, and his result, in agreement with [28], shows that this friction component is

very small compared with the orthogonal components but not exactly zero. In this section, we consider the most general case in which the axial component is included. In order to do that, the following vorticity tensor  $\mathbf{P}_\omega$  is used [17]:

$$\mathbf{P}_\omega^R = \frac{1}{2}\kappa L_R \left[ (B - B'') (\mathbf{U} - \hat{\Omega}\hat{\Omega}) + B'\mathbf{W} \cdot \hat{\Omega} + 2B''\hat{\Omega}\hat{\Omega} \right], \quad (3.1)$$

where  $\mathbf{U}$  is the unit matrix,  $\mathbf{W}$  the Ricci tensor, and  $B$  and  $B'$  are the Hall-Vinen coefficients [26] describing the orthogonal dissipative and non dissipative contributions while  $B''$  is the friction coefficient along the rotational axis. Using the Eq. (3.1), the production term in (2.1d) can be expressed as [4, 17]:

$$\vec{\sigma}_\omega^R = \frac{1}{2}\kappa L_R \left[ (B - B'')\hat{\Omega} \wedge (\hat{\Omega} \wedge \mathbf{q}) + B'\hat{\Omega} \wedge \mathbf{q} - 2B''\hat{\Omega}\hat{\Omega} \cdot \mathbf{q} \right]. \quad (3.2)$$

The interest to consider spatial distribution of vortices and anisotropy of mutual friction in rotating container has led Mathieu et al. in [29] to analyze a more general case in which a parallelepipedic cavity filled of helium II rotates around an axis tilted an angle  $\theta$  with respect to its wall. In the following Subsection we will show that the results of the latter experiments can be easily explained using the general expression (3.2).

Substituting the expression (3.2) into the system (2.1) and choosing  $\hat{\Omega} = (\Omega, 0, 0)$ , the system assumes the following form:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_i}{\partial x_i} = 0 \\ \rho \frac{\partial v_i}{\partial t} + \frac{\partial p}{\partial x_i} + 2\rho\Omega v_j \epsilon_{1ji} = 0 \\ \frac{\partial T}{\partial t} + \frac{1}{\rho c_V} \frac{\partial q_i}{\partial x_i} = 0 \\ \frac{\partial q_i}{\partial t} + \zeta \frac{\partial T}{\partial x_i} + (2\Omega - \frac{1}{2}B'\kappa L_R) q_j \epsilon_{1ji} = \frac{1}{2}\kappa L_R [(B - B'')(-q_i + q_1 \delta_{i1}) - 2B''q_1 \delta_{i1}], \end{cases} \quad (3.3)$$

where  $\epsilon_{kji}$  is the Ricci tensor.

It is easily observed that a stationary solution of this system is:

$$\rho = \rho_0, \quad \mathbf{v} = \mathbf{0}, \quad T = T_0, \quad \mathbf{q} = \mathbf{0}. \quad (3.4)$$

In order to study the propagation of plane harmonic waves of small amplitude [30], we put  $\Gamma = (\rho, v_i, T, q_i)$ , and we look for solutions of the linearized system of field equations (2.1) of the form:

$$\Gamma = \Gamma_0 + \tilde{\Gamma} e^{i(K n_j x_j - \omega t)}, \quad (3.5)$$

where  $\Gamma_0 = (\rho_0, 0, T_0, 0)$  denotes the unperturbed state,  $\tilde{\Gamma} = (\tilde{\rho}, \tilde{v}_i, \tilde{T}, \tilde{q}_i)$  small amplitudes whose products can be neglected,  $K = k_r + ik_s$  is the wavenumber,  $\omega = \omega_r + i\omega_s$  the frequency and  $\mathbf{n} = (n_i)$  the unit vector orthogonal to the wave front. Along this paper we will assume that the propagating waves do not affect the vortex tangle, i.e. that they do not contribute to the production nor the destruction of vortices. In other terms, the waves are used to explore a given vortex tangle, without modifying it. If the wave amplitude is high enough, it could yield new contributions to the tangle.

In the following we assume that  $\Omega$  is small, so that the term  $\mathbf{i}_0$  in (2.1b) can be neglected. For the sake of simplicity, the subscript 0, which denotes quantities referring to the unperturbed state  $\Gamma_0$ , will be dropped out.

### 3.1 First case: $\mathbf{n}$ parallel to $\Omega$

In this subsection we analyze the case in which the unit vector  $\mathbf{n}$  orthogonal to the wave front is parallel to the axis of rotation, i.e.  $\mathbf{n} = (1, 0, 0)$ . Substituting (3.5) into the linearized system (3.3) and letting  $\mathbf{t}_1 = (0, 0, 1)$  and  $\mathbf{t}_2 = (0, 1, 0)$  as unit vectors tangent to the wave front, the following homogeneous algebraic linear system for the small amplitudes is obtained:

$$\left\{ \begin{array}{l} -\omega \tilde{\rho} + \rho K \tilde{v}_1 = 0 \\ -\omega \tilde{v}_1 + K \frac{\rho}{\rho} \tilde{\rho} = 0 \\ -\omega \tilde{T} + \frac{K}{\rho c_V} \tilde{q}_1 = 0 \\ (-\omega - iB''\kappa L_R) \tilde{q}_1 + \zeta K \tilde{T} = 0 \\ \\ -\omega \tilde{v}_3 - 2i\Omega \tilde{v}_2 = 0 \\ -\omega \tilde{v}_2 + 2i\Omega \tilde{v}_3 = 0 \\ \\ \left( -\omega - \frac{i}{2}\kappa L_R (B - B'') \right) \tilde{q}_3 - i \left( 2\Omega - \frac{1}{2}\kappa L_R B' \right) \tilde{q}_2 = 0 \\ \left( -\omega - \frac{i}{2}\kappa L_R (B - B'') \right) \tilde{q}_2 + i \left( 2\Omega - \frac{1}{2}\kappa L_R B' \right) \tilde{q}_3 = 0. \end{array} \right. \quad (3.6)$$

From the above system, it follows that longitudinal and transversal modes evolve independently. The study of the longitudinal modes furnishes the existence of two waves: the first is known as *first sound* or *pressure wave* in which density and velocity vibrate, and the second is known as *second sound* or *temperature wave* in which temperature and heat flux vibrate. Therefore, as observed in [27], when the wave is propagated parallel to the rotation axis, the longitudinal modes are influenced by the rotation only through the axial component of the mutual friction ( $B''$  coefficient). In fact, the first two equations of the system (3.6), give for the first sound  $V_1 := \frac{\omega}{K} = \sqrt{p_\rho}$ , whereas the third and fourth equation, with the assumption  $K = k_r + ik_s$  and  $\omega$  real, give second sound waves with the following velocity and attenuation:

$$w_2 := \frac{\omega}{k_r} = \sqrt{\frac{4V_2^4 k_r^2}{4V_2^2 k_r^2 + B''^2 \kappa L_R^2}} \quad \text{and} \quad k_s = \frac{w_2 B'' \kappa L_R}{2V_2^2} \quad (3.7)$$

where  $V_2^2 := \frac{\zeta}{\rho c_V}$  is the velocity of the second sound in the absence of vortices. Therefore, the following fields vibrate respectively:

$\omega_{1,2} = \pm k V_1$	$\omega_{3,4} = \pm \sqrt{\frac{4V_2^4 k_r^4}{4V_2^2 k_r^2 + B''^2 \kappa L_R^2}}$
$\tilde{\rho} = \psi$ $\tilde{v}_1 = \pm \frac{V_1}{\rho} \psi$ $\tilde{T}_0 = 0$ $\tilde{q}_1 = 0$	$\tilde{\rho} = 0$ $\tilde{v}_1 = 0$ $\tilde{T} = T_0 \psi$ $\tilde{q}_1 = \pm \rho c_V T_0 \sqrt{\frac{4V_2^4 k_r^4}{4V_2^2 k_r^2 + B''^2 \kappa L_R^2}} \psi$

On the contrary, the transversal modes are influenced by the rotation. In fact, by considering the fifth and the sixth equation of (3.6) they admit nontrivial solutions if and only if its determinant vanishes; this yields  $\omega_{5,6} = \pm 2|\Omega|$ .

Now, we consider the equations seven and eight of the system (3.6) and, as above, we find the following dispersion relation:

$$\left(2\Omega - \frac{1}{2}\kappa L_R B'\right)^2 - \left(-\omega - \frac{i}{2}\kappa L_R(B - B'')\right)^2 = 0, \quad (3.8)$$

whose solutions are

$$\omega_{7,8} = \pm(2\Omega - \frac{1}{2}\kappa L_R B') - \frac{i}{2}\kappa L_R(B - B''). \quad (3.9)$$

These transversal modes are influenced from both dissipative and nondissipative contributions  $B$ ,  $B'$  and  $B''$  in the interaction between quasi-particles and vortex lines.

### 3.2 Second case: $\mathbf{n}$ orthogonal to $\Omega$

In this subsection we assume that the direction of propagation of the waves is orthogonal to the rotation axis, i.e. for example,  $\mathbf{n} = (0, 1, 0)$ . The unit vectors tangent to the wave front are  $\mathbf{t}_1 = (1, 0, 0)$  and  $\mathbf{t}_2 = (0, 0, 1)$ . Under these assumptions, substituting (3.5) into the linearized system (3.3), the following system is obtained:

$$\begin{cases} -\omega\tilde{\rho} + \rho K\tilde{v}_2 = 0 \\ -\omega\tilde{v}_2 + K\frac{\rho}{\rho}\tilde{\rho} + 2i\Omega\tilde{v}_3 = 0 \\ -\omega\tilde{v}_3 - 2i\Omega\tilde{v}_2 = 0 \\ -\omega\tilde{T} + \frac{K}{\rho c_V}\tilde{q}_2 = 0 \\ (-\omega - \frac{i}{2}\kappa L_R(B - B''))\tilde{q}_2 + \zeta K\tilde{T} + i(2\Omega - \frac{1}{2}\kappa L_R B')\tilde{q}_3 = 0 \\ (-\omega - \frac{i}{2}\kappa L_R(B - B''))\tilde{q}_3 - i(2\Omega - \frac{1}{2}\kappa L_R B')\tilde{q}_2 = 0 \\ -\omega\tilde{v}_1 = 0 \\ (-\omega - iB''\kappa L_R)\tilde{q}_1 = 0. \end{cases} \quad (3.10)$$

In this case, the longitudinal and transversal modes do not evolve independently. The first sound is coupled with one of the two transversal modes in which velocity vibrates; while the second sound is coupled with a transversal mode in which heat flux vibrates.

Studying the first three equations of the system (3.10), we obtain a dispersion relation whose solutions are:

$$\omega_1 = 0, \quad (3.11)$$

$$\omega_{2,3} = \pm V_1 \sqrt{\left(1 - 4\frac{\Omega^2}{\omega_{2,3}^2}\right)^{-1}}. \quad (3.12)$$

Summarizing:

$\omega_1 = 0$	$\omega_{2,3} \simeq \pm KV_1 + O(\Omega^2)$
$\tilde{\rho} = \psi$	$\tilde{\rho} = \psi$
$\tilde{v}_2 = 0$	$\tilde{v}_2 = \frac{\pm V_1}{\rho}\psi$
$\tilde{v}_3 = i\frac{KV_1^2}{2\Omega\rho}\psi$	$\tilde{v}_3 = -\frac{2i\Omega}{\rho K}\psi$

The second three equations admit non trivial solutions if and only if their determinant vanishes. Neglecting the second-order terms in  $\Omega$ , the dispersion relation becomes:

$$\left(-\omega - \frac{i}{2}\kappa L_R(B - B'')\right) \left[-\omega \left(-\omega - \frac{i}{2}\kappa L_R(B - B'')\right) - K^2 V_2^2\right] = 0 \quad (3.13)$$

For  $\omega \in \Re$  and  $K = k_r + ik_s$  complex, one gets the solution  $\omega_4 = 0$ , which represents a stationary mode; and two solutions which furnish the following phase velocity and attenuation coefficient of the temperature wave:

$$w_2^2 := \frac{\omega^2}{k_r^2} = V_2^2 \frac{2}{1 + \sqrt{1 + \frac{(B-B'')^2 \kappa^2 L_R^2}{4\omega^2}}}, \quad (3.14)$$

$$k_s = \frac{(B - B'')\kappa L_R w_2}{4V_2^2}. \quad (3.15)$$

The approximated solutions to second order in  $\frac{(B-B'')\kappa L_R}{\omega}$  are:

$$w_2 \simeq V_2 \left(1 - \frac{(B - B'')^2 \kappa^2 L_R^2}{32\omega^2}\right) + O\left(\frac{(B - B'')^4 \kappa^4 L_R^4}{\omega^4}\right), \quad (3.16)$$

$$k_s \simeq \frac{(B - B'')\kappa L_R}{4V_2} + O\left(\frac{(B - B'')^3 \kappa^3 L_R^3}{\omega^2}\right) \quad (3.17)$$

Summarizing, when the direction of propagation of the waves is orthogonal to the rotation axis, the temperature wave experiences a strong attenuation, which grows with  $\Omega$ . The corresponding modes are:

$\omega_4 = 0$	$\omega_{5,6} \simeq \pm k_r V_2 \left(1 - \frac{(B-B'')^2 \kappa^2 L_R^2}{32\omega^2}\right)$
$\tilde{T} = -\frac{i(2\Omega - \frac{1}{2}\kappa L_R B')}{\zeta K} \psi$	$\tilde{T} = T_0 \psi$
$\tilde{q}_2 = 0$	$\tilde{q}_2 = \frac{T_0 \zeta}{V_2} \left(1 - \frac{(B-B'')^2 \kappa^2 L_R^2}{32\omega^2}\right) \psi$
$\tilde{q}_3 = \psi$	$\tilde{q}_3 = \frac{i(2\Omega - \frac{1}{2}\kappa L_R B') T_0 \zeta \left(1 - \frac{(B-B'')^2 \kappa^2 L_R^2}{32\omega^2}\right)}{V_2 \left[\pm k_r V_2 \left(1 - \frac{(B-B'')^2 \kappa^2 L_R^2}{32\omega^2}\right) - \frac{i}{2}(B-B'')\kappa L_R\right]} \psi$

We note that in the mode  $\omega_4 = 0$ , only the transversal component of the heat flux is involved. For  $\omega = \omega_r + i\omega_s$  complex and  $K \in \Re$ , the solutions of dispersion relation (3.13) are:

$$\begin{aligned} \omega_4 &= -\frac{i}{2}(B - B'')\kappa L_R, \\ \omega_{5,6} &= \pm \sqrt{K^2 V_2^2 - \frac{1}{16}(B - B'')^2 \kappa^2 L_R^2} - i \frac{(B - B'')\kappa L_R}{4}. \end{aligned}$$

The first mode, with  $\omega_4 = -\frac{i}{2}(B - B'')\kappa L_R$ , corresponds to an extremely slow relaxation phenomenon involving the temperature wave and the transversal component of the heat flux:

$\omega_4 = -\frac{i}{2}(B - B'')\kappa L_R$
$\tilde{T} = -\frac{i(2\Omega - \frac{1}{2}\kappa L_R B')}{\zeta K} \psi$
$\tilde{q}_2 = 0$
$\tilde{q}_3 = \psi$



which when  $\Omega \rightarrow 0$ , converges to a stationary mode. The attenuation in  $\omega_{5,6}$  (corresponding to  $q_1$  and  $q_2$ ) is physically reasonable in view of (3.2), where it is seen that for  $\mathbf{q}$  parallel to  $\mathbf{\Omega}$  the only component of the friction force is the axial one (related to the coefficient  $B''$ ), whereas for  $\mathbf{q}$  orthogonal to the vortex line (i.e. to  $\mathbf{\Omega}$ ) there is an attenuation dependent on the dissipative coefficient  $(B - B'')$ .

## 4 Wave propagation in presence of thermal counterflow

In this section, we study wave propagation in presence of pure thermal counterflow in liquid Helium II to compare the results with those of [14] and those of Section 5. Let us consider a flow channel that connects two He II reservoirs (as shown in fig. 1). When a steady heat is applied to one end of the channel, there exists a temperature difference  $\Delta T$  between the two ends. From the microscopic point of view using the two-fluid model, since only the normal fluid component carries entropy and heat flow, it will move away from the heat source (left reservoir) to the right reservoir and then give up the heat. At the same time, the superfluid component must counter-flow from right to left to conserve the mass. When it arrives at the left reservoir, part of the superfluid component will be converted to normal fluid by absorbing heat. Thus, a relative counterflow between the normal fluid and superfluid components is established, and this internal convection process is termed thermal counterflow, which is associated to the heat flux  $\mathbf{q}$  through the relation  $\mathbf{q} = \rho_s T s \mathbf{V}$ .

In this case, assuming that the vortex tangle caused by the counterflow is isotropic, the vorticity tensor  $\mathbf{P}_\omega$ , as indicated in [14], takes the following form:

$$\mathbf{P}_\omega^H = \frac{1}{3} \kappa B L \mathbf{U} \quad \Rightarrow \quad \vec{\sigma}_\omega^H = -\frac{1}{3} \kappa B L \mathbf{q}, \quad (4.1)$$

where  $L = \gamma^2 q^2$ . Under this assumption, the linearized set of field equations read as:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_j}{\partial x_j} = 0 \\ \rho \frac{\partial v_i}{\partial t} + \frac{\partial p}{\partial x_i} = 0 \\ \frac{\partial T}{\partial t} + \frac{1}{\rho c_V} \frac{\partial q_j}{\partial x_j} = 0 \\ \frac{\partial q_i}{\partial t} + \zeta \frac{\partial T}{\partial x_i} = -\frac{1}{3} \kappa B L q_i \end{cases} \quad (4.2)$$

A stationary solution of the system (4.2) is [14]:

$$\rho = \rho_0, \quad \dot{\mathbf{v}} = \mathbf{0}, \quad T = T(x) = T_0 - \frac{\kappa B L}{3\zeta} q_0 x, \quad \mathbf{q} = \mathbf{q}_0 \quad (4.3)$$

where  $x$  is the direction of the heat flux  $\mathbf{q} = \mathbf{q}_0$ . In order to study the propagation of harmonic plane waves in the channel, we look for solutions of the system (4.2) of the form:

$$\Gamma = \Gamma_0 + \tilde{\Gamma} e^{i(K n_j x_j - \omega t)}, \quad (4.4)$$

where  $\Gamma_0 = (\rho_0, 0, T(x), \mathbf{q}_0)$ , and the following homogeneous algebraic linear system for the small amplitudes is obtained:

$$\begin{cases} -\omega \tilde{\rho} + \rho K \tilde{v}_j n_j = 0 \\ -\rho \omega \tilde{v}_i + p_\rho K \tilde{\rho} n_i = 0 \\ -\omega \tilde{T} + \frac{K}{\rho c_V} \tilde{q}_j n_j = 0 \\ (\omega + \frac{1}{3} i \kappa B L) \tilde{q}_i - \zeta K \tilde{T} n_i = 0. \end{cases} \quad (4.5)$$

The longitudinal modes are obtained projecting the vectorial equations for the small amplitudes of velocity and heat flux on the direction orthogonal to the wave front. It is observed that the first sound is not influenced by the thermal counterflow, while the velocity and the attenuation of the second sound are influenced by the presence of the vortex tangle. The results are:

$$w_1 = \pm \sqrt{p_\rho}$$

with  $p_\rho$  standing for  $\partial p / \partial \rho$  and:

$$\begin{aligned} w_2 &= V_2 \sqrt{\left(1 + \frac{k_s^2 V_2^2}{\omega^2}\right)^{-1}} \Rightarrow w_2 \simeq V_2 \left(1 - k_s^2 \frac{V_2^2}{2\omega^2}\right), \\ k_s &= \frac{1}{6} \kappa B L w_2. \end{aligned}$$

These results generalize those of [14] where the terms in  $k_s^2$  have been neglected.

The transversal modes are obtained projecting the vectorial equations for the small amplitudes of velocity and heat flux on the wave front, obtaining:

$$\begin{cases} -\omega \tilde{v}_\pi = 0 \\ (-\omega + \frac{i}{3} \kappa B L) \tilde{q}_\pi = 0 \end{cases} \quad (4.6)$$

where  $\pi$  denotes the tangential plane to the wave front. The solutions of this equation are:  $\omega_1 = 0$  and  $\omega_2 = \frac{i}{3} \kappa B L$ . The first mode ( $\omega_1 = 0$ ) is a stationary mode.

## 5 Wave propagation with simultaneous rotation and counterflow

The combined situation of rotation and heat flux (as shown in fig. 2), is a relatively new area of research [16]-[21]. The first motivation of this great interest is that from the experimental observations one deduces that the two effects are not merely additive; in particular, for  $\mathbf{q}$  or  $\mathbf{\Omega}$  high, the measured values of  $L$  are always less than  $L_H + L_R$ .

Under the simultaneous influence of thermal counterflow  $V$  and rotation speed  $\Omega$ , rotation produces an ordered array of vortex lines parallel to rotation axis, whereas counterflow velocity causes a disordered tangle. In this way the total vortex line is given by the superposition of both contributions so that the vortex tangle is anisotropic [17], [18]. Therefore, assuming that the rotation is along the  $x$  direction  $\mathbf{\Omega} = (\Omega, 0, 0)$  and isotropy in the transversal ( $y-z$ ) plane, for the vorticity tensor  $\mathbf{P}_\omega$ , in combined situation of counterflow and rotation, the following explicit expression is taken:

$$\mathbf{P}_\omega = \gamma \kappa L \left\{ \frac{2}{3} (1 - D) \mathbf{U} + D \left[ \left(1 - \frac{B''}{B}\right) (\mathbf{U} - \hat{\mathbf{\Omega}} \hat{\mathbf{\Omega}}) + \frac{B'}{B} \mathbf{W} \cdot \hat{\mathbf{\Omega}} + 2 \frac{B''}{B} \hat{\mathbf{\Omega}} \hat{\mathbf{\Omega}} \right] \right\} \quad (5.1)$$

where  $\gamma$  is linked to the coefficient  $B$  through the relation  $\gamma = B/2$  and  $D$  is a parameter between 0 and 1 related to the anisotropy of vortex lines, describing the relative weight of the array of vortex lines parallel to  $\mathbf{\Omega}$  and the disordered tangle of counterflow (when  $D = 0$  we recover an isotropic tangle – Eq. (4.1), whereas when  $D = 1$  the ordered array – Eq. (3.1)). Assuming  $b = \frac{1}{3}(1 - D) + \frac{DB''}{B}$  and  $c = \frac{B'D}{B}$ , the vorticity tensor (5.1) can be written as:

$$\mathbf{P}_\omega = \gamma \kappa L \left\{ \begin{pmatrix} 2b & 0 & 0 \\ 0 & 1-b & 0 \\ 0 & 0 & 1-b \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix} \right\}. \quad (5.2)$$

Note that the isotropy in the  $y-z$  plane may only be assumed when both  $\mathbf{\Omega}$  and  $\mathbf{V}$  are directed along the  $x$  axis. A more general situations is yet an open topic.

Substituting the expression (5.2) into the linearized set of field equations (2.1), it assumes the following form:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_j}{\partial x_j} = 0 \\ \rho \frac{\partial v_i}{\partial t} + \frac{\partial p}{\partial x_i} + 2\rho \Omega v_j \epsilon_{1ji} = 0 \\ \frac{\partial T}{\partial t} + \frac{1}{\rho c_V} \frac{\partial q_j}{\partial x_j} = 0 \\ \frac{\partial q_i}{\partial t} + \zeta \frac{\partial T}{\partial x_i} + 2\Omega q_j \epsilon_{1ji} = -\gamma \kappa L \{ 2b q_1 \delta_{1i} + [(1-b) q_2 + c q_3] \delta_{2i} + [(1-b) q_3 - c q_2] \delta_{3i} \} \end{cases} \quad (5.3)$$

A stationary solution of this system is:

$$\begin{aligned} \rho &= \rho_0, \quad \dot{\mathbf{v}} = \mathbf{0}, \quad \mathbf{q} = \mathbf{q}_0 \equiv (q_0, 0, 0) \\ T &= T(x_i) = T_0 - 2 \frac{\gamma \kappa L}{\zeta} b q_0 \delta_{1i} x_i. \end{aligned}$$

In order to study the propagation of harmonic plane waves, we look for solutions of (5.3) of the following form:

$$\Gamma = \Gamma_0 + \tilde{\Gamma} e^{i(K n_j x_j - \omega t)}, \quad (5.4)$$

where  $\Gamma_0 = (\rho_0, 0, T(x_i), \mathbf{q}_0)$  and  $T(x_i)$  is a linear function of  $x_i$ .

Now, we investigate two different cases:  $\mathbf{n}$  parallel to  $\mathbf{\Omega}$  and  $\mathbf{n}$  orthogonal to  $\mathbf{\Omega}$ ; the latter is the only case for which experimental data exist [6].

### 5.1 First case: $\mathbf{n}$ parallel to $\mathbf{\Omega}$

In this subsection we analyze the case in which the unit vector  $\mathbf{n}$  orthogonal to the wave front is parallel to the direction of the rotation, i.e.  $\mathbf{n} = (1, 0, 0)$ . Letting  $\mathbf{t}_1 = (0, 1, 0)$  and  $\mathbf{t}_2 = (0, 0, 1)$  as unit vectors tangent to the wave front, the system (5.3) for the small amplitudes (5.4) is:

$$\begin{cases} -\omega \tilde{\rho} + K \rho \tilde{v}_1 = 0 \\ -\omega \tilde{v}_1 + K \frac{p_\rho}{\rho} \tilde{\rho} = 0 \\ -\omega \tilde{T} + \frac{K}{\rho c_V} \tilde{q}_1 = 0 \\ [-\omega - 2i\gamma \kappa L b] \tilde{q}_1 + \zeta K \tilde{T} = 0 \\ \\ -\omega \tilde{v}_2 + 2i\Omega \tilde{v}_3 = 0 \\ -\omega \tilde{v}_3 - 2i\Omega \tilde{v}_2 = 0 \\ \\ [-\omega - i\gamma \kappa L (1-b)] \tilde{q}_2 + (2i\Omega - i\gamma \kappa L c) \tilde{q}_3 = 0 \\ [-\omega - i\gamma \kappa L (1-b)] \tilde{q}_3 - (2i\Omega - i\gamma \kappa L c) \tilde{q}_2 = 0 \end{cases} \quad (5.5)$$

In this case the longitudinal and transversal modes evolve independently. In particular, we can observe that the first sound, given by the study of the first two equations of the system (5.5), is not influenced by the presence of the vortex tangle:

$$\begin{array}{|l} \omega_{1,2} = \pm k_r V_1 \\ \hline \tilde{\rho} = \psi \\ \tilde{v}_1 = \frac{V_1}{\rho} \psi \end{array}$$

whereas the second sound suffers extra attenuation due to the vortex tangle. The third and fourth equation of the system (5.5) admit non trivial solutions if and only if their determinant vanishes, obtaining in this way the following dispersion relation:

$$\omega^2 + 2i\gamma\kappa Lb\omega - K^2V_2^2 = 0. \quad (5.6)$$

Supposing that  $\omega$  is real and  $K = k_r + ik_s$  is complex, the dispersion relation admits the solutions:

$$w_2^2 := \frac{\omega^2}{k_r^2} = V_2^2 \frac{2}{1 + \sqrt{1 + \frac{4\gamma^2\kappa^2L^2b^2}{\omega^2}}}, \quad (5.7)$$

$$k_s = \frac{\gamma\kappa Lbw_2}{V_2^2}. \quad (5.8)$$

When  $\Omega = 0$  and  $b = 1/3$  the results of the Section 4 are obtained. The approximate solutions to second order in  $\frac{\gamma\kappa Lb}{\omega}$  are:

$$w_2 \simeq V_2 \left( 1 - \frac{\gamma^2\kappa^2L^2b^2}{2\omega^2} \right) + O\left( \frac{\gamma^4\kappa^4L^4b^4}{\omega^4} \right), \quad (5.9)$$

$$k_s \simeq \frac{\gamma\kappa Lb}{V_2} + O\left( \frac{\gamma^3\kappa^3L^3b^3}{\omega^2} \right). \quad (5.10)$$

Now, we study the transversal modes. The second subsystem (fifth and sixth equation) of the system (5.5) admits nontrivial solutions if and only if its determinant vanishes; this yields:

$$\omega^2 - 4\Omega^2 = 0. \quad (5.11)$$

The solutions of this equation are  $\omega_{5,6} = \pm 2|\Omega|$ . The respective modes are:

$$\left| \begin{array}{l} \omega_{5,6} = \pm 2|\Omega| \\ \hline \tilde{v}_3 = \psi \\ \tilde{v}_2 = \pm i\psi \end{array} \right|$$

and they correspond to extremely slow phenomena, which, when  $\Omega \rightarrow 0$ , tend to stationary modes.

Finally, we consider the last subsystem (equations seven and eight), whose dispersion relation is:

$$\omega^2 + 2i\gamma\kappa L(1-b)\omega + \left[ -(\gamma\kappa L(1-b))^2 - 4\Omega^2 + 4\gamma\Omega\kappa Lc - (\gamma\kappa Lc)^2 \right] = 0 \quad (5.12)$$

which admits the following exact solutions:

$$\omega_{7,8} = \pm (2\Omega - \gamma\kappa Lc) - i\gamma\kappa L(1-b). \quad (5.13)$$

The corresponding modes are:

$$\left| \begin{array}{l} \omega_{7,8} = \pm (2\Omega - \gamma\kappa Lc) - i\gamma\kappa L(1-b) \\ \hline \tilde{q}_3 = \psi \\ \tilde{q}_2 = \pm i\psi \end{array} \right|$$

From (5.7), (5.8) and (5.13) one may obtain the following quantities  $L$ ,  $b$  and  $c$ :

$$L = \frac{-\omega_s w_2 + V_2^2 k_s}{\gamma \kappa w_2}, \quad b = \frac{V_2^2 k_s}{-\omega_s w_2 + V_2^2 k_s}, \quad c = \frac{-\omega_r w_2 + 2\Omega w_2}{-\omega_s w_2 + V_2^2 k_s} \quad (5.14)$$

where we have put  $\omega_7 = \omega_r + i\omega_s$ .

The results of this section, from the physical point of view, imply that measurement in a single direction are enough to give information on all the variables describing the vortex tangle.

## 5.2 Second case: $\mathbf{n}$ orthogonal to $\Omega$

Now we assume that the direction of propagation of the waves is orthogonal to the rotation axis, i.e. for example,  $\mathbf{n} = (0, 1, 0)$ . The unit vectors tangent to the wave front are  $\mathbf{t}_1 = (1, 0, 0)$  and  $\mathbf{t}_2 = (0, 0, 1)$ . Under these assumptions, the homogeneous algebraic linear system for the small amplitudes is:

$$\left\{ \begin{array}{l} -\omega \tilde{\rho} + K \rho \tilde{v}_2 = 0 \\ -\omega \tilde{v}_2 + K \frac{p_\rho}{\rho} \tilde{\rho} + 2i\Omega \tilde{v}_3 = 0 \\ -\omega \tilde{v}_3 - 2i\Omega \tilde{v}_2 = 0 \\ \\ -\omega \tilde{T} + \frac{K}{\rho c_V} \tilde{q}_2 = 0 \\ -\omega \tilde{q}_2 + \zeta K \tilde{T} + 2i\Omega \tilde{q}_3 = i\gamma \kappa L [(1-b)\tilde{q}_2 + c\tilde{q}_3] \\ -\omega \tilde{q}_3 - 2i\Omega \tilde{q}_2 = i\gamma \kappa L [(1-b)\tilde{q}_3 - c\tilde{q}_2] \\ \\ -\omega \tilde{v}_1 = 0 \\ [-\omega - i\gamma \kappa 2Lb] \tilde{q}_1 = 0 \end{array} \right. \quad (5.15)$$

In this case the longitudinal and the transversal modes not evolve independently. In particular, the first sound is coupled with one of the two transversal modes in which velocity vibrates, while the second sound is coupled with a transversal mode in which heat flux vibrates.

As in the previous subsection, the first subsystem (first three equations) of the system (5.15), admits non trivial solutions if and only if its determinant vanishes:

$$-\omega [\omega^2 - 4\Omega^2 - K^2 p_\rho] = 0. \quad (5.16)$$

The solutions of this equation, as also the corresponding modes, are the same to the case of pure rotation (see equations (3.11)-(3.12)).

The second subsystem (fourth and fifth equations), has the dispersion relation:

$$(-\omega - i\gamma \kappa L(1-b)) [\omega (-\omega - i\gamma \kappa L(1-b)) + K^2 V_2^2] + \omega (2i\Omega - i\gamma \kappa Lc)^2 = 0. \quad (5.17)$$

Assuming  $\omega \in \Re$  and  $K = k_r + ik_s$ , one obtains the following two equations:

$$\begin{aligned} -\omega^3 + \gamma^2 \kappa^2 L^2 (1-b)^2 \omega + 4\Omega^2 \omega + \gamma^2 \kappa^2 L^2 c^2 \omega - 4\gamma \kappa L c \Omega \omega + \\ + k_r^2 V_2^2 \omega - k_s^2 V_2^2 \omega - 2\gamma \kappa L (1-b) k_r k_s V_2^2 = 0, \end{aligned} \quad (5.18)$$

$$-2\gamma \kappa L (1-b) \omega^2 + 2k_r k_s V_2^2 \omega + \gamma \kappa L (1-b) (k_r^2 - k_s^2) V_2^2 = 0. \quad (5.19)$$

In the hypothesis of small dissipation ( $k_r^2 \gg k_s^2$ ), from (5.19) one obtains:

$$k_s = \gamma \kappa L (1 - b) \left( \frac{2w_2^2 - V_2^2}{2w_2 V_2^2} \right), \quad (5.20)$$

which substituting in (5.18), yields:

$$\omega^4 - \left[ (2\Omega - \gamma \kappa L c)^2 - \gamma^2 \kappa^2 L^2 (1 - b)^2 \right] \omega^2 - k_r^2 V_2^2 \omega^2 - \gamma^2 \kappa^2 L^2 (1 - b)^2 V_2^2 k_r^2 = 0. \quad (5.21)$$

Putting  $\tilde{A} = - \left[ (2\Omega - \gamma \kappa L c)^2 - \gamma^2 \kappa^2 L^2 (1 - b)^2 \right]$  and  $\tilde{B} = -\gamma^2 \kappa^2 L^2 (1 - b)^2$  and taking into account that  $w_2 = \frac{\omega}{k_r}$ , the Eq. (5.21) becomes:

$$w_2^2 \left[ w_2^2 \left( 1 + \frac{\tilde{A}}{\omega^2} \right) - V_2^2 \left( 1 - \frac{\tilde{B}}{\omega^2} \right) \right] = 0 \quad (5.22)$$

whose solutions are:

$$w_2^2 = 0, \quad \text{and} \quad w_2^2 = V_2^2 \frac{(\omega^2 - \tilde{B})}{(\omega^2 + \tilde{A})} = V_2^2 \frac{1}{1 - \frac{(2\Omega - \gamma \kappa L c)^2}{\omega^2 + \gamma^2 \kappa^2 L^2 (1 - b)^2}}. \quad (5.23)$$

We can remark that the coefficients  $\tilde{A}$  and  $\tilde{B}$  are negative and that  $w_2^2 \geq V_2^2$  because  $\omega^2 + \tilde{A} \leq \omega^2 - \tilde{B}$  and, in particular,  $w_2^2 = V_2^2$  for  $\Omega = \frac{\gamma \kappa L c}{2}$ . Now, studying the transversal modes, i.e. the third subsystem (equations seventh and eighth), we obtain  $\omega_7 = 0$ , which corresponds to a stationary mode, and:

$$\omega_8 = -i\gamma \kappa 2Lb. \quad (5.24)$$

Summarizing, also in this case measurement in a single direction are enough to given information on all the variables describing the vortex tangle, namely  $L$ ,  $b$  and  $c$ , from equations (5.20), (5.23) and (5.24):

$$\begin{aligned} L &= \frac{4k_s w_2 V_2^2 - \omega_s (2w_2 - V_2^2)}{2(2w_2^2 - V_2^2) \gamma \kappa}, \\ b &= -\frac{\omega_s (2w_2^2 - V_2^2)}{4k_s w_2 V_2^2 - \omega_s (2w_2 - V_2^2)}, \\ c &= \frac{4\Omega(2w_2^2 - V_2^2) - \sqrt{(1 - V_2^2)(4k_r^2(2w_2^2 - V_2^2)^2 + 16k_s^2 V_2^4)}}{4k_s w_2 V_2^2 - \omega_s (2w_2^2 - V_2^2)} \end{aligned} \quad (5.25)$$

where we have put  $\omega_8 = i\omega_s$  and  $\omega_s = 2\gamma \kappa L B$ .

In this subsection we have analyzed wave propagation in the combined situation of rotation and counterflow with the direction  $n$  orthogonal to  $\Omega$ . In [6] Swanson et al. experimented the same situation, but they didn't represent the attenuation neither the speed of the second sound but only the vortex line density  $L$  as function of  $\Omega$  and  $V$ . Therefore, it is unknown how they plotted these graphics, which the hypothesis were made and what was the anisotropy considered. Instead, the results of these two subsections allow to know the spatial distribution of the vortex tangle simply by performing experiments on waves propagating orthogonally to  $\Omega$  (equations (5.14)) or parallelly to  $\Omega$  (equations (5.25)).

## 6 Conclusions

In this work we have studied the propagation of waves (longitudinal density and temperature waves, longitudinal and transversal velocity and heat waves) in turbulent superfluid helium in the three situations: rotating frame, thermal counterflow, and simultaneous thermal counterflow and rotation.

From the physical point of view it is interesting to note that our detailed analysis in Section 5 shows that, in contrast to which one could intuitively expect, measurements in a single direction are enough to give information on all the variables describing the vortex tangle, namely  $L$ ,  $b$  and  $c$ , for instance, from one of (5.7)-(5.8) and (5.13) or of (5.20)-(5.23) and (5.24). This is not an immediate intuitive result. Future analyses of work along this direction could be, for instance, to consider that  $\mathbf{\Omega}$  and  $\mathbf{V}$  have arbitrary directions, i.e. that they are not parallel to each other, in which case (5.2) would not be sufficient to describe the vortex tangle, because no isotropy in the  $y - z$  plane could be assumed.

Another topic could be to assume that the external waves produce vibrations in the vortex lines, without creating nor destroying them. An example of that is the work of Barenghi et al. [28]. A more general possibility would be to consider that nonlinear effects of the external waves create and destroy new vortices. Yet another topic would be to consider what happens with waves whose wavevector  $\lambda$  become short enough to be comparable with the average vortex separation, of the order  $L^{-1/2}$ . In this case, one could study nonlocal effects in the vortex [33, 34]. The first mentioned application could be carried out within the existing physical model, at the expenses of more cumbersome calculations. In contrast, the other three applications need more progress in the basic physical understanding of the problem.

## Acknowledgments

We acknowledge Prof. D. Jou (Departament de Física, Universitat Autònoma de Barcelona) and Prof. M.S. Mongiovì (Dipartimento di Metodi e Modelli Matematici, Università di Palermo) for the enlightening discussions useful to the deepening of these arguments. This work is supported by MIUR of Italy under "Azioni Integrate Italia-Spagna, anno 2005". R.A. P. is supported by an "Assegno di ricerca dell'Istituto Nazionale di Alta Matematica 'F. Severi' (INdAM)" of Italy and M.S. is supported by an "Assegno di ricerca MIUR" of Italy.

## References

- [1] L. Tisza, Nature 141 (1938) 913.
- [2] L.D. Landau, J.Phys. 5 (1941) 71.
- [3] J.T. Tough, Prog. Low Temp. Phys. 8 (1982) 133.
- [4] R.J. Donnelly, Quantized Vortices in Helium II, Cambridge University Press, Cambridge, UK, 1991.
- [5] R.T. Wang, C.E. Swanson, R.J. Donnelly, Phys.Rev.B 36 (1987) 5240.
- [6] C.E Swanson, C.F. Barenghi, R.J. Donnelly, Phys.Rev.Lett. 50 (1983) 190.
- [7] M.S. Mongiovì, Phys. Rev. B 48 (1993) 6276.

- [8] D. Jou, J. Casas-Vázquez, G. Lebon, *Extended Irreversible Thermodynamics*, Springer-Verlag, Berlin, 2001.
- [9] I. Müller, T. Ruggeri, *Rational Extended Thermodynamics*, Springer-Verlag Berlin, 1998.
- [10] M.S. Mongiovì, R.A. Peruzza, *Zangew.Math.Phys.* 54 (2003) 566.
- [11] M.S. Mongiovì, R.A. Peruzza, *I.J.Nonlinear Mech.* 39 (2004) 1005.
- [12] M.S. Mongiovì, R.A. Peruzza, *Math. Comp. Model.* 38 (2003) 409.
- [13] M.S. Mongiovì, R.A. Peruzza, D. Jou, *Recent research developments in Physics, Transworld Research Network*, (2004), 1033.
- [14] D. Jou, G. Lebon, M.S. Mongiovì, *Phys. Rev. B* 66 (2002) 224509.
- [15] C. F. Barenghi, R.J. Donnelly, W.F. Vinen, *Quantized Vortex Dynamics and Superfluid Turbulence*, Springer-Berlin, 2001.
- [16] D. Jou, M.S. Mongiovì, *Phys. Rev. B* 69 (2004) 094513.
- [17] D. Jou, M.S. Mongiovì, *Phys. Rev. B* 72 (2005) 144517.
- [18] D. Jou, M.S. Mongiovì, *Phys. Rev. B* 74 (2006) 054509.
- [19] M.S. Mongiovì, D. Jou, *Phys. Rev. B* 72 (2005) 104515.
- [20] M. Tsubota, C.F. Barenghi, T. Araki, A. Mitani, *Phys. Rev. B* 69 (2004) 134515.
- [21] M. Tsubota, T. Araki, C.F. Barenghi, *Jour. Low Temp. Phys.* 134 (2004) 471.
- [22] M.S. Mongiovì, *Physica A*, 291, (2001) 518.
- [23] M.S. Mongiovì D. Jou, *Phys. Rev. B*, 75, (2007) 024507.
- [24] D. Jou, M.S. Mongiovì, M. Sciacca, "Vortex density waves in a hydrodynamical model of superfluid turbulence", *Phys. Lett. A*, in press
- [25] K.L. Henderson and C.F. Barenghi, *Theoret. Comput. Fluid Dynam.* 18 (2004) 183.
- [26] H.E. Hall, W. F. Vinen, *Proc. R. Soc. London A*238 (1956) 215; *Proc. Roy. Soc. A*238 (1956) 204.
- [27] H.A. Snyder, *Physics of Fluids* 6 (1963) 755.
- [28] C.F. Barenghi, M. Tsubota, A. Mitani, T. Araki, *J. Low Temp. Phys.* 134 (2004) 489.
- [29] P. Mathieu, B. Plaçais, Y. Simon, *Phys. Rev. B* 29 (1984) 2489.
- [30] J. Whitham, *Linear and Nonlinear Waves*, New York, Wiley, 1974.
- [31] I.M. Khalatnikov, *An Introduction to the Theory of Superfluidity*, New York, Benjamin, 1965.
- [32] Y.A. Sergeev, C.F. Barenghi, *J. Low. Tem. Phys.* 127 (2002) 203.
- [33] M.S. Mongiovì, D. Jou, *Phys. Rev. B* 71 (2005) 094507.
- [34] M.S. Mongiovì, D. Jou, *J. Phys: Condens. Matter* 17 (2005) 4423.



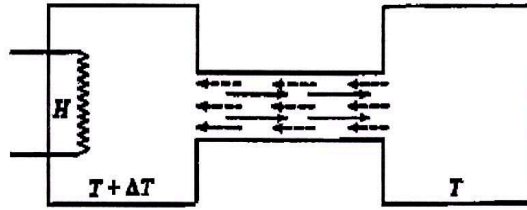


Fig. 1. Counterflow container configuration.

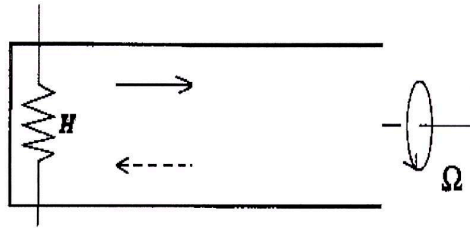


Fig. 2. Rotating counterflow container configuration.